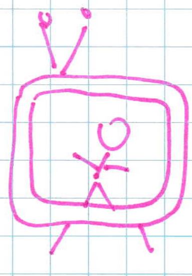
6. Logarithmes et exponentielles

Préambule



$x + 4 = 7 \Rightarrow x = 7 - 4 = 3$

$x \cdot 4 = 12 \Rightarrow x = \frac{12}{4} = 3$

$x^4 = 81 \Rightarrow x = \sqrt[4]{81} = 3$

$3^x = 81 \Rightarrow x = \log_3(81) = 4$

Exercice 6.1 (Réviser d'abord le chapitre 4)

a. $\log(1) = 0$ car $10^0 = 1$

b. $\log(10^7) = 7$ car $10^7 = 10^7$

c. $\log\left(\frac{1}{10}\right) = -1$ car $10^{-1} = \frac{1}{10}$

d. $\log\left(\frac{1}{\sqrt[4]{10}}\right) = -\frac{1}{4}$ car $10^{-\frac{1}{4}} = \frac{1}{\sqrt[4]{10}}$

e. $\log(\sqrt[3]{100}) = \log(\sqrt[3]{10^2}) = \frac{2}{3}$, car $10^{\frac{2}{3}} = \sqrt[3]{10^2}$

f. $\log(-10)$ n'existe pas

g. $\log(x)$ quand $x \to 0$, $x > 0 \Rightarrow \log(x) \to -\infty$

Exercice 6.2

a. $\log_2(8) = 3$, car $2^3 = 8$

b. $\log_2(64) = 6$, car $2^6 = 64$

c. $\log_3(729) = 6$, car $3^6 = 729$

d. $\log_9(729) = 3$, car $9^3 = 729$

e. $\log_3(\sqrt[4]{\underset{\uparrow\ 3^3}{27}}) = \frac{3}{4}$, car $3^{\frac{3}{4}} = \sqrt[4]{3^3}$

f. $\log_2\left(\frac{1}{4}\right) = -2$, car $2^{-2} = \frac{1}{4}$

g. $\log_3\left(\frac{1}{81}\right) = -4$, car $3^{-4} = \frac{1}{81}$

h. $\log_2\left(\frac{1}{128}\right) = -7$, car $2^{-7} = \frac{1}{128}$

i. $\log_{16}\left(\frac{1}{2}\right) = -\frac{1}{4}$, car $16^{-\frac{1}{4}} = \frac{1}{2}$

j. $\log_9\left(\frac{1}{3}\right) = -\frac{1}{2}$, car $9^{-\frac{1}{2}} = \frac{1}{3}$

Exercice 6.3

a. $x = 5$, car $2^5 = 32$

b. $\log_4(4^3) = \log_4(64) = 3 \Rightarrow x = 64$

c. $\log_x(125) = 3 \Rightarrow x^3 = 125 \Rightarrow x = 5$

Exercice 6.4

a. $\log(9) \cong 0{,}954$

b. $10^{0,9} \cong 7{,}94$

$10^{0,95} = 8{,}91$

$10^{0,96} = 9{,}12$

$10^{0,955} = 9{,}01$

$10^{0,954} \cong 8{,}995 \quad \Rightarrow$ réponse à 3 décimales : 0,954

$10^{0,9545} = 9{,}005 \quad \leftarrow$ on continue pour le fun...

$10^{0,9542} = 8{,}999$ etc.

c. On laisse tomber... Boum!

Exercice 6.5

a. $\log(1270) \cong 3{,}104$

$\log(127) \cong 2{,}104$ $\quad -1$

$\log(12{,}7) = 1{,}104$ $\quad -1$

$\log(1{,}27) = 0{,}104$

$\log(0{,}127) = -0{,}896$ $\quad -1$

$\log(0{,}0127) = -1{,}896$

b. $\log\left(\frac{x}{10}\right) = \log(x) - 1$

c. c'est la propriété 5a, avec $v = 10$. Or $\log(10) = 1$. Donc on retrouve la formule du point b.

Exercice 6.6

a. $\log(a^2b^3) = \log(a^2) + \log(b^3) = 2\log(a) + 3\log(b)$

b. $\log\left(\frac{a^3}{b^2}\right) = \log(a^3) - \log(b^2) = 3\log(a) - 2\log(b)$

c. $\log(a) + \log(\underset{d^{\frac{1}{2}}}{\sqrt{d}}) - \log(c) - \log(\underset{b^{\frac{1}{3}}}{\sqrt[3]{b}}) = \log(a) - \frac{1}{3}\log(b) - \log(c) + \frac{1}{2}\log(d)$

Exercice 6.7

a. $\ln(e) = 1$, car $e^1 = e$ b. $\ln(1) = 0$, car $e^0 = 1$

c. $\ln(e^7) = 7$ d. $\ln\left(\frac{1}{e}\right) = -1$, car $e^{-1} = \frac{1}{e}$

e. $\ln(\sqrt{e}) = \frac{1}{2}$, car $e^{\frac{1}{2}} = \sqrt{e}$ f. $\ln\left(\frac{1}{\sqrt{e}}\right) = -\frac{1}{2}$, car $e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$

(même principe que les ex 6.1 et 6.2, mais dans la base e)

Exercice 6.8

Méthode : utiliser les propriétés du § 6.3 pour se ramener à une équation du type $\log_b(\text{machin}) = \log_b(\text{truc})$. On pourra ensuite supprimer les $\log_b$ et résoudre avec des méthodes vues en début d'année.

a. $\log(x+1) - \log(3) = \log(2x-3) + \log(7)$

pr. 5a) $\log\left(\frac{x+1}{3}\right) = \log(7\cdot(2x-3))$ pr 4a.

On peut supprimer les log, car ils sont les deux dans la même base et on bien $\log(\text{machin}) = \log(\text{truc})$

$\Rightarrow \frac{x+1}{3} = 14x - 21$

$x + 1 = 42x - 63$

$41x = 64 \quad \Rightarrow x = \frac{64}{41}$

⚠ Il faut vérifier dans l'équation de départ que la solution est correcte. Il peut arriver que la solution trouvée soit une solution fantôme. On a déjà vu ce phénomène avec les équations irrationnelles.

$\log\left(\frac{64}{41}+1\right) - \log(3) \overset{?}{=} \log\left(2\cdot\frac{64}{41}-3\right) + \log(7)$

$0,4084 - 0,4771 = -0,9138 + 0,845$ ✓

b. $\log_3(2x-5) + \log_3(3x+7) = 4\log_3(2)$

pr. 4a

$\log_3((2x-5)(3x+7)) = \log_3(2^4)$ pr 6a

$(2x-5)(3x+7) = 16$

$6x^2 - x - 35 = 16$

$6x^2 - x - 51 = 0$

$x_{1,2} = \dfrac{1 \pm \sqrt{1+24\cdot 51}}{12}$ $\begin{cases} \underline{\underline{3}} \\ \cancel{-2,83} \end{cases}$

On voit tout de suite que ça ne marche pas. On aurait le log d'un nombre négatif, ce qui est impossible.

Vérification pour $x=3$:

$\underbrace{\log_3(1)}_{0} + \log_3(16) \overset{?}{=} 4\cdot\log_3(2)$

$\log_3(16) = \log_3(2^4)$ ✓

c. $\ln(x^2-7) = 2\ln(x+3)$

$\ln(x^2-7) = \ln((x+3)^2)$

$\cancel{x^2} - 7 = \cancel{x^2} + 6x + 9$

$-16 = 6x$

$x = \underline{\underline{-\frac{8}{3}}}$

Vérification

$\ln\left(\left(-\frac{8}{3}\right)^2 - 7\right) \overset{?}{=} 2\ln\left(-\frac{8}{3}+3\right)$

$-2,197 = -2,197$ ✓

d. $\log(\sqrt{x+1}) + \log(\sqrt{x-1}) = \log(5)$

pr. 4a

$\log(\sqrt{x+1}\sqrt{x-1}) = \log(5)$

$\sqrt{(x+1)(x-1)} = 5 \quad (\,)^2$

$x^2 - 1 = 25 \Rightarrow x = \begin{cases} \underline{\underline{\sqrt{26}}} \\ \cancel{-\sqrt{26}} \end{cases}$ clairement impossible

Vérification pour $x = \sqrt{26}$

$\log(\sqrt{\sqrt{26}+1}) + \log(\sqrt{\sqrt{26}-1}) \overset{?}{=} \log(5)$

$0,3926 + 0,3063 = 0,6989$ ✓

e. $\log(x^2+3x-1) = 2$ | $10^{()}$ des deux côtés

pr. 3b $\underbrace{10^{\log(x^2+3x-1)}}$ $= 10^2$ | on simplifie

$x^2+3x-1 = 100$

$x^2+3x-101 = 0$

$x_{1,2} = \dfrac{-3 \pm \sqrt{9+404}}{2} < \begin{matrix} \underline{\underline{8,6612}} \\ \underline{\underline{-11,6612}} \end{matrix}$

Vérification pour $8,6612$: $\log(8,6612^2+3\cdot 8,6612-1) = 2$ ✓

" " $-11,6612$: $\log((-11,6612)^2-3\cdot 11,6612-1) = 2$ ✓

Exercice 6.9

a. $3^x + 9^x = 90$ $\qquad 9 = 3^2 \Rightarrow 9^x = (3^2)^x = 3^{2x} = (3^x)^2$

$y + y^2 = 90$

$y^2 + y - 90 = 0$

$y_{1,2} = \dfrac{-1 \pm \sqrt{1+360}}{2} < \begin{matrix} 9 \\ -10 \end{matrix}$ $\qquad$ ⚠ on cherche x, pas y

Comme $y = 3^x$, on a $3^x = 9 \Rightarrow \underline{\underline{x = 2}}$.

" " on a $3^x = -10$, ce qui est impossible $(3^x > 0, \forall x)$

b. $e^{3x} = 5$

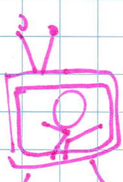
pr. 3a $\underbrace{\ln(e^{3x})}$ $= \ln(5)$

$3x = \ln(5)$

$x = \dfrac{\ln(5)}{3} \cong 0,5365$

c. $4e^{-3x} - 3e^{-x} - e^x = 0$ $\qquad | \cdot e^{-x}$

$4e^{-4x} - 3\cdot e^{-2x} - 1 = 0$ $\qquad y = e^{-2x}$ $\qquad$ oui je sais, ce n'est pas évident...



$4y^2 - 3y - 1 = 0$

$y_{1,2} = \dfrac{3 \pm \sqrt{9+16}}{8} < \begin{matrix} 1 \\ -\frac{1}{4} \end{matrix}$ $\qquad$ ⚠ on cherche x, pas y

Comme $y = e^{-2x}$, on a $e^{-2x} = 1 \Rightarrow \underline{\underline{x = 0}}$

" " , on a $e^{-2x} = -\frac{1}{4}$, ce qui est impossible ($e^{-2x} > 0, \forall x$)

Exercice 6.10

a. $2^{x^2} = 512$

$2^{x^2} = 2^9$

$x^2 = 9 \Rightarrow \underline{\underline{x = \pm 3}}$

b. $7^{x^2+x} = 49$

$7^{x^2+x} = 7^2$

$x^2 + x = 2$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$\Rightarrow \underline{\underline{x_1 = -2}}, \ \underline{\underline{x_2 = 1}}$

c. $\frac{1}{10^x} = 10'000$

$10^{-x} = 10^4$

$-x = 4 \Rightarrow \underline{\underline{x = -4}}$

Ex 6.11

a. $\begin{cases} \log(x) + \log(y) = 2 \\ x + y = 25 \end{cases}$

$\log(x) + \log(y) = 2 \Rightarrow \log(x \cdot y) = 2 \Rightarrow \underbrace{10^{\log(x\cdot y)}}_{x \cdot y} = 10^2$

$x \cdot y = 100$

$x + y = 25 \Rightarrow x = 25 - y$

$\Rightarrow (25 - y)y = 100$

$-y^2 + 25y - 100 = 0$

$y^2 - 25y + 100 = 0$

$(y-20)(y-5) = 0 \Rightarrow y_1 = 20 \qquad y_2 = 5$

$x_1 = 5 \qquad x_2 = 20$

b. $\begin{cases} \ln(x) - \ln(y) = 1 \\ x \cdot y = e \Rightarrow x = \frac{e}{y} \end{cases}$

$\ln\left(\frac{e}{y}\right) - \ln(y) = 1$

$\ln\left(\frac{e}{y^2}\right) = 1 \Rightarrow$ il faut que $\frac{e}{y^2} = e \Rightarrow y = \begin{cases} 1 \\ \cancel{-1} \end{cases}$

puisque $\ln(e) = 1$

car $\ln(-1)$ n'existe pas

$\Rightarrow \underline{\underline{y = 1}} \Rightarrow \underline{\underline{x = e}}$

Exercice 6.12

$$\log_2(16) = \frac{\log(16)}{\log(2)} = 4 = \frac{\ln(16)}{\ln(2)}$$ (on a le choix)

$$\log_3(16) = \frac{\log(16)}{\log(3)} \cong 2,524$$

$$\log_4(16) = \frac{\log(16)}{\log(4)} = 2$$ etc. Je pense que vous avez compris...

$\vdots$

Exercice 6.13

Cet exercice est le plus dur du chapitre. Si vous le maîtrisez, cela veut dire que vous avez tout compris ! Il ressemble à l'ex 6.8, sauf que les log ne sont pas forcément dans la même base.

a. $\log(x) - \log(x+1) = 3\log(4)$

pr. 5a $\quad \log\left(\frac{x}{x+1}\right) = \log(4^3) \quad$ pr. 6a

$$\frac{x}{x+1} = 64$$

$$x = 64(x+1)$$

$$x = 64x + 64 \Rightarrow x = \cancel{-\frac{64}{63}}$$ impossible car $\log\left(-\frac{64}{63}\right)$ n'existe pas

b. $\log_2(x+7) + \log_2(x) = 3$

pr 4a $\quad \log_2((x+7)\cdot x) = \log_2(2^3) \quad$ pr. 3a

$$x^2 + 7x = 8$$

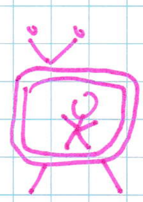
$$x^2 + 7x - 8 = 0$$

$$(x+8)(x-1) = 0 \Rightarrow x = \begin{cases} \cancel{-8} & \text{car } \log_2(-8) \text{ n'existe pas} \\ \underline{\underline{1}} \end{cases}$$

Vérification pour $x = 1$: $\log_2(8) + \log_2(1) \overset{?}{=} 3$

$$3 + 0 = 3 \quad ✓$$

Exercice 6.13 c, façon exercice de style



$$\log_3(x) = \frac{1}{2} + \log_9(4x+15)$$

En base 3

$$\log_3(x) = \log_3\left(3^{\frac{1}{2}}\right) + \frac{\log_3(4x+15)}{\underbrace{\log_3(9)}_{=2}}$$

$$\log_3(x) = \log_3(\sqrt{3}) + \frac{1}{2}\log_3(4x+15)$$

$$\log_3(x) = \log_3(\sqrt{3}) + \log_3\left((4x+15)^{\frac{1}{2}}\right)$$

$$\log_3(x) = \log_3\left(\sqrt{3}\cdot\sqrt{4x+15}\right) = \log_3\left(\sqrt{12x+45}\right)$$

$$x = \sqrt{12x+45} \qquad |(\ )^2$$

$$x^2 = 12x+45$$

☆ $$x^2 - 12x - 45 = 0$$

$$x_{1,2} = \frac{12 \pm \sqrt{144+4\cdot 45}}{2} = \frac{12\pm\sqrt{324}}{2} = \frac{12\pm 18}{2} \quad \begin{cases} \underline{\underline{15}} \\ \cancel{-3} \end{cases}$$

vérification: $\log_3(15) \overset{?}{=} \frac{1}{2} + \log_9(75)$

$$\frac{\log(15)}{\log(3)} \overset{?}{=} \frac{1}{2} + \frac{\log(75)}{\log(9)} \Rightarrow 2{,}4649 = 0{,}5 + 1{,}9649 \quad ✓$$

En base 9

$$\frac{\log_9(x)}{\underbrace{\log_9(3)}_{\frac{1}{2}}} = \log_9\left(9^{\frac{1}{2}}\right) + \log_9(4x+15)$$

$$2\cdot\log_9(x) = \log_9(3) + \log_9(4x+15)$$

$$\log_9(x^2) = \log_9(12x+45)$$

$$x^2 = 12x+45$$

☆ ... (on a déjà fait tous les calculs un peu plus haut)

En base 10

$$\log_3(x) = \frac{1}{2} + \log_9(4x+15)$$

$$\frac{\log(x)}{\log(3)} = \frac{1}{2} + \frac{\log(4x+15)}{\log(9)} \qquad \Big| \cdot \log(9)$$

$$\frac{\log(9)}{\log(3)} \cdot \log(x) = \frac{1}{2} \cdot \log(9) + \log(4x+15)$$

$$\frac{\log(3^2)}{\log(3)} = \frac{2\cancel{\log(3)}}{\cancel{\log(3)}} = 2 \qquad \log(9^{\frac{1}{2}}) = \log(3)$$

$$2\log(x) = \log(3) + \log(4x+15)$$

$$\log(x^2) = \log(12x+45)$$

$$x^2 = 12x+45$$

☆ ...

En base e

$$\frac{\ln(x)}{\ln(3)} = \frac{1}{2} + \frac{\ln(4x+15)}{\ln(9)} \qquad \Big| \ln(9)$$

$$\frac{\ln(9)}{\ln(3)} \cdot \ln(x) = \frac{1}{2} \cdot \ln(9) + \ln(4x+15)$$

$$\frac{\ln(3^2)}{\ln(3)} = \frac{2\cancel{\ln(3)}}{\cancel{\ln(3)}} = 2 \qquad \ln(9^{\frac{1}{2}}) = \ln(3)$$

$$2\ln(x) = \ln(3) + \ln(4x+15)$$

$$\ln(x^2) = \ln(12x+45)$$

$$x^2 = 12x+45$$

☆ ...

d. $\log_4(x) = \frac{1}{8}\log_2(x^2+2)$

$\frac{\log_2(x)}{\log_2(4)} = \log_2\left((x^2+2)^{\frac{1}{8}}\right)$; $2 = \log_2(4)$

$\frac{1}{2}\log_2(x) = \log_2\left((x^2+2)^{\frac{1}{8}}\right)$

$\log_2(x^{\frac{1}{2}}) = \log_2\left((x^2+2)^{\frac{1}{8}}\right)$

$x^{\frac{1}{2}} = (x^2+2)^{\frac{1}{8}} \quad \mid (\ )^8$

$x^4 = x^2+2$

bicarrée → $x^4 - x^2 - 2 = 0 \quad \mid y = x^2$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0 \Rightarrow y_1 = 2 \Rightarrow x_1 = \sqrt{2}$; ~~$x_2 = -\sqrt{2}$~~

impossible à cause du $\log_4(x)$

$y_2 = -1 \Rightarrow$ pas de solutions pour x_3 et x_4

Vérification

$\underbrace{\log_4(\sqrt{2})}_{\frac{1}{4}} = \frac{1}{8}\underbrace{\log_2(4)}_{2}$ ✓

$\underbrace{\log_4(1)}_{0} = \frac{1}{8}\underbrace{\log_2(3)}_{\neq 0}$ ✗

e. $\log_4(x) = -3 + \log_2(x+16)$

$\frac{\log_2(x)}{\log_2(4)} = \log_2(2^{-3}) + \log_2(x+16)$; $2 = \log_2(4)$

$\frac{1}{2}\log_2(x) = \log_2\left(\frac{1}{8}\right) + \log_2(x+16)$

$\log_2(x^{\frac{1}{2}}) = \log_2\left(\frac{1}{8}(x+16)\right)$

$\sqrt{x} = \frac{1}{8}x + 2 \quad \mid (\ )^2$

ne pas oublier ! $(a+b)^2 = a^2 + \underline{\underline{2ab}} + b^2$

$x = \frac{1}{64}x^2 + \frac{1}{2}x + 4 \quad \mid \cdot 64$

$64x = x^2 + 32x + 256$

$x^2 - 32x + 256 = 0$ identité remarquable

$(x-16)^2 = 0$

$\Rightarrow x = 16$

Vérifions : $\log_4(16) \overset{?}{=} -3 + \log_2(32)$

$2 = -3 + 5$ ✓

f. $\log_3(x) \cdot \log_9(x) = 2$

$$\log_3(x) \cdot \frac{\log_3(x)}{\underbrace{\log_3(9)}_{=2}} = 2$$

$$\frac{1}{2} \cdot (\log_3(x))^2 = 2$$

$$(\log_3(x))^2 = 4 \Rightarrow \begin{cases} \log_3(x) = 2 \Rightarrow \overbrace{3^{\log_3(x)}}^{=x} = 3^2 = 9 \\ \log_3(x) = -2 \Rightarrow \underbrace{3^{\log_3(x)}}_{=x} = 3^{-2} = \frac{1}{9} \end{cases}$$

Vérifions : $\underbrace{\log_3(9)}_{2} \cdot \underbrace{\log_9(9)}_{1} = 2$ ✓

$\underbrace{\log_3(\frac{1}{9})}_{-2} \cdot \underbrace{\log_9(\frac{1}{9})}_{-1} = 2$ ✓

$\underline{\underline{x_1 = 9}}$, $\underline{\underline{x_2 = \frac{1}{9}}}$

g. $\log(x^2) = \log^2(x)$

$2 \cdot \log(x) = \log^2(x)$ | Posons $y = \log(x)$

$2y = y^2$

$y^2 - 2y = 0$

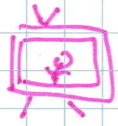
$y(y-2) = 0 \Rightarrow y = 0 \Rightarrow \log(x) = 0 \Rightarrow \underline{\underline{x_1 = 1}}$

$\qquad\qquad\quad y = 2 \Rightarrow \log(x) = 2 \Rightarrow \underline{\underline{x_2 = 100}}$

Vérifions :

$\log(1^2) = \log^2(1) \Rightarrow 0 = 0^2$ ✓

$\log(100^2) = \log^2(100) \Rightarrow 4 = 2^2$ ✓

Exercice 6.14  avec DESMOS

Observez les différences avec les courbes de $\ln(x)$ et e^x.

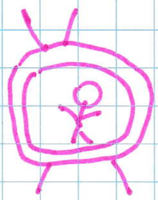
$\ln(x-1)$: décalage de 1 vers la droite ⚠ contre intuitif

$\ln(x+1)$: " " " " la gauche

$\ln(x)+1$: " " " vers le haut

$\ln(x)-1$: " " " vers le bas.

Exercice 6.15



$$N(t) = N_0\, e^{-\mu \cdot t}$$

a. $\cancel{5}\cdot\cancel{10^{10}} = \cancel{5}\cdot\cancel{10^{11}}\, e^{-1,2\cdot 10^{-4}\cdot t}$

$$1 = 10 \cdot e^{-1,2\cdot 10^{-4} t}$$

$$\frac{1}{10} = e^{-1,2\cdot 10^{-4} t} \qquad \mid \ln(\,)$$

$$\ln(0,1) = \underbrace{\ln(e^{-1,2\cdot 10^{-4} t})}$$

$$\ln(0,1) = -1,2\cdot 10^{-4}\, t$$

$$t = \frac{\ln(0,1)}{-1,2\cdot 10^{-4}} = \frac{-2,3}{-1,2\cdot 10^{-4}} = 1,92\cdot 10^{4} = \underline{\underline{19'200 \text{ ans}}}$$

b. $\cancel{2,5\cdot 10^{11}} = \cancel{5\cdot 10^{11}}\, e^{-1,2\cdot 10^{-4} t}$

$$\frac{1}{2} = e^{-1,2\cdot 10^{-4} t} \qquad \mid \ln(\,)$$

$$\ln(0,5) = \underbrace{\ln(e^{-1,2\cdot 10^{-4} t})}$$

$$\ln(0,5) = -1,2\cdot 10^{-4} t$$

$$t = \frac{\ln(0,5)}{-1,2\cdot 10^{-4}} = 0,5776\cdot 10^{4} = \underline{\underline{5776 \text{ ans}}}$$

Exercice 6.16

$$N(t) = N_0 \cdot 0,805^{t}$$

⚠ t est donné en millénaires

$t = 0,1 \Rightarrow 100$ ans $\Rightarrow 1$ siècle

nombre de mots qui restent après t millénaires.

a. $N(0,1) = N_0 \cdot 0,805^{0,1}$

$N(0,1) = 0,978\, N_0$ $\Rightarrow$ Il reste 97,8% des mots après 1 siècle

Donc 2,2% des mots ont disparu.

c. $\frac{1}{4}\cancel{N_0} = \cancel{N_0}\, 0,805^{t}$

il reste $\frac{1}{4}$ des mots

marche aussi avec $\ln(\,)$

$$\log(0,25) = \log(0,805^{t}) = t \cdot \log(0,805)$$

$$\Rightarrow t = \frac{\log(0,25)}{\log(0,805)} \simeq 6,4 \text{ millénaires, donc } \underline{\underline{6400 \text{ ans}}}.$$

Exercice 6.17

$P(h) = P_0 \cdot e^{-\alpha h}$ ⚠ h en km.

a. $P(2) = 10^5 \cdot e^{-0{,}125 \cdot 2} = 0{,}7788 \cdot 10^5 = \underline{\underline{77\,880 \text{ Pa}}}$

b. $6 \cdot \cancel{10^4} = 10^{\cancel{5}} \cdot e^{-0{,}125 \cdot h}$

$0{,}6 = e^{-0{,}125 h}$

$\ln(0{,}6) = \underbrace{\ln(e^{-0{,}125 h})}$

$-0{,}5108 = -0{,}125 h \Rightarrow h \cong 4{,}1 \text{ km} \cong \underline{\underline{4000 \text{ mètres}}}$

Exercice 6.18

a. $I = I_0 \cdot c^x$

$\log(I) = \log(I_0 \cdot c^x)$

$\log(I) = \log(I_0) + \log(c^x)$

$\log(I) = \log(I_0) + x \cdot \log(c) \Rightarrow x = \underline{\underline{\dfrac{\log(I) - \log(I_0)}{\log(c)}}}$

b. $x = \dfrac{\log(0{,}01 I_0) - \log(I_0)}{\log(0{,}25)}$

$= \dfrac{\log\left(\frac{0{,}01 I_0}{I_0}\right)}{\log(0{,}25)} = \dfrac{\log(0{,}01)}{\log(0{,}25)} = \underline{\underline{3{,}32 \text{ m}}}$

Exercice 6.19

$h_1 = 12 \leftrightarrow M_1 = 6{,}3 = \log\left(\dfrac{12}{y_0}\right) \Rightarrow 10^{6{,}3} = 10^{\log\left(\frac{12}{y_0}\right)} = \dfrac{12}{y_0}$

$h_2 = 3{,}6 \leftrightarrow M_2 = \log\left(\dfrac{3{,}6}{y_0}\right)$ $\Rightarrow y_0 = \dfrac{12}{10^{6{,}3}}$

$= \log\left(3{,}6 \cdot \dfrac{10^{6{,}3}}{12}\right) = \log(598\,578{,}6986) \cong \underline{\underline{5{,}8}}$

$h_3 = 1{,}1 \leftrightarrow M_3 = \log\left(1{,}1 \cdot \dfrac{10^{6{,}3}}{12}\right) \cong \underline{\underline{5{,}3}}$

$h_4 = 1{,}0 \leftrightarrow M_4 = \log\left(1{,}0 \cdot \dfrac{10^{6{,}3}}{12}\right) \cong \underline{\underline{5{,}2}}$

Exercice 6.20

$10^{()}$ des 2 côtés

a. $\log(E) = 4{,}4 + 1{,}5 \cdot 7{,}2 = 15{,}2 \Rightarrow E = 10^{15,2} = \underline{\underline{1{,}585 \cdot 10^{15}}}$

b. $\log(E) = 4{,}4 + 1{,}5 \cdot 9{,}1 = 18{,}05 \Rightarrow E = 10^{18,05} = \underline{\underline{1{,}122 \cdot 10^{18}}}$

c. $\log(E) = 4{,}4 + 1{,}5 \cdot 4{,}2 = 10{,}7 \Rightarrow E = 10^{10,7} = \underline{\underline{5{,}012 \cdot 10^{10}}}$

d. $\log(E_7) = 4{,}4 + 1{,}5 \cdot 7 = 14{,}9 \Rightarrow E_7 = 10^{14,9}$

$\log(E_8) = 4{,}4 + 1{,}5 \cdot 8 = 16{,}4 \Rightarrow E_8 = 10^{16,4}$

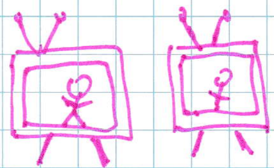
$$\Rightarrow \frac{E_8}{E_7} = \frac{10^{16,4}}{10^{14,9}} = 10^{16,4-14,9} = 10^{1,5} = \underline{\underline{31{,}62}}$$

e. $E = 15 \cdot 10^6 \cdot 4{,}2 \cdot 10^6 = 63 \cdot 10^{12} = 6{,}3 \cdot 10^{13}$

$$\log(6{,}3 \cdot 10^{13}) = 4{,}4 + 1{,}5 \cdot M$$

$$\log(6{,}3) + \underbrace{\log(10^{13})}_{=13} = 4{,}4 + 1{,}5\,M \Rightarrow M = \frac{\log(6{,}3) + 13 - 4{,}4}{1{,}5} = \underline{\underline{6{,}27}}$$

Exercice 6.21



$N(t) = N_0\, e^{\beta \cdot t}$ $\qquad N_0$ et β inconnus.

Après 3 jours : $N(3) = 2 \cdot 10^5$ $\qquad$ Après 4,5 jours : $N(4{,}5) = 16 \cdot 10^5$

$2 \cdot 10^5 = N_0 \cdot e^{\beta \cdot 3} \Rightarrow N_0 = \dfrac{2 \cdot 10^5}{e^{3\beta}}$

$16 \cdot 10^5 = N_0 \cdot e^{\beta \cdot 4,5} \Rightarrow N_0 = \dfrac{16 \cdot 10^5}{e^{4,5\beta}}$

$$\frac{\cancel{2}^{1} \cdot \cancel{10^5}}{e^{3\beta}} = \frac{\cancel{16}^{8} \cdot \cancel{10^5}}{e^{4,5\beta}}$$

$$e^{4,5\beta} = 8 \cdot e^{3\beta}$$

$$\frac{e^{4,5\beta}}{e^{3\beta}} = \boxed{8 = e^{1,5\beta}} \Rightarrow \ln(8) = \underbrace{\ln(e^{1,5\beta})}_{1,5\beta} \Rightarrow \beta = \frac{\ln(8)}{1{,}5} \cong \underline{\underline{1{,}3863}}$$

$$\Rightarrow N_0 = \frac{2 \cdot 10^5}{e^{3 \cdot 1,3863}} = \frac{2 \cdot 10^5}{64} = \underline{\underline{3125}}$$

a. $N(5) = 3125 \cdot e^{1,3863 \cdot 5} = 32 \cdot 10^5 = 3,2 \cdot 10^6$

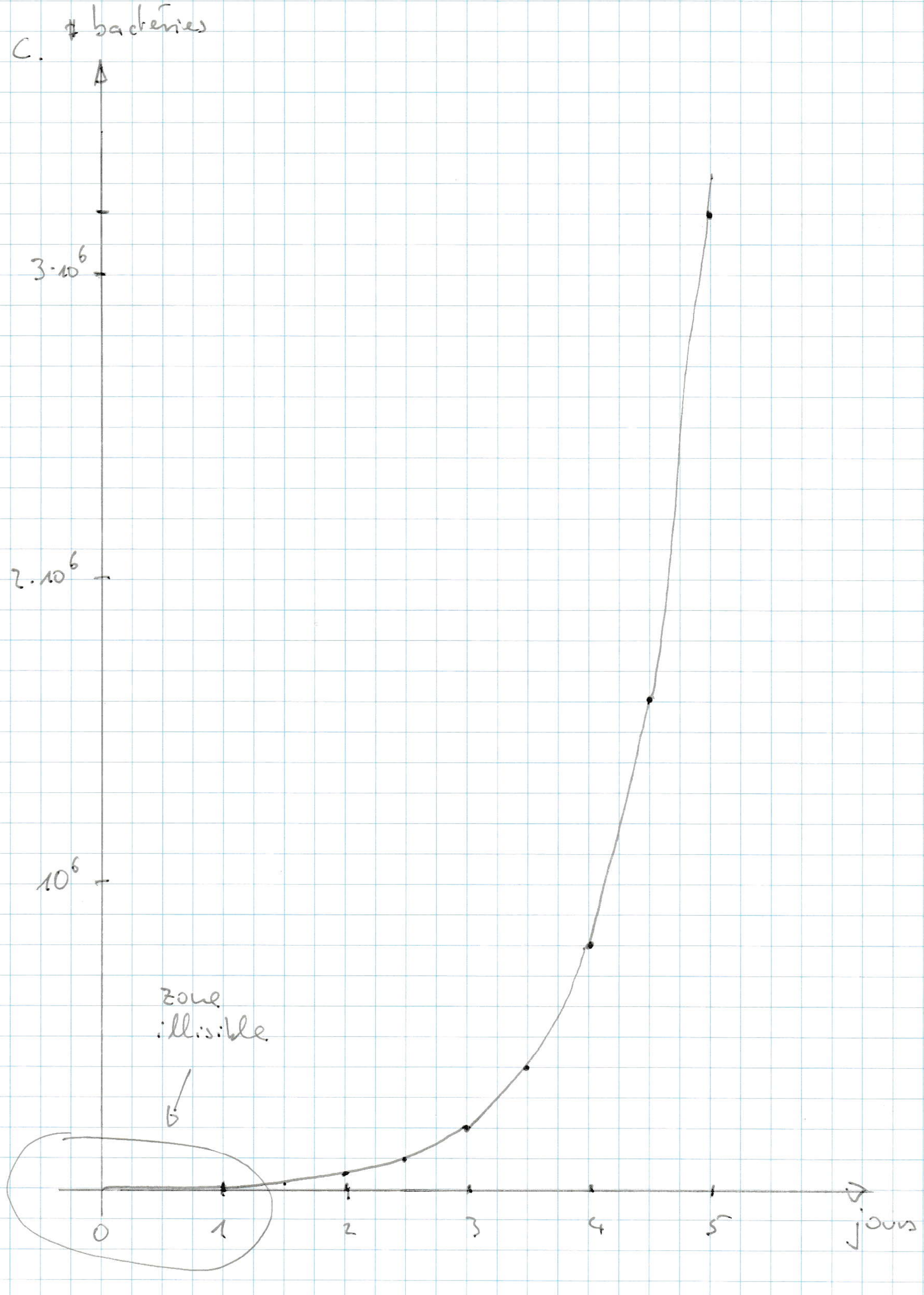
b. $8 \cdot 10^5 = 3125 \cdot e^{1,3863 \cdot t}$

$$\frac{8 \cdot 10^5}{3125} = e^{1,3863\, t}$$

$$256 = e^{1,3863\, t}$$

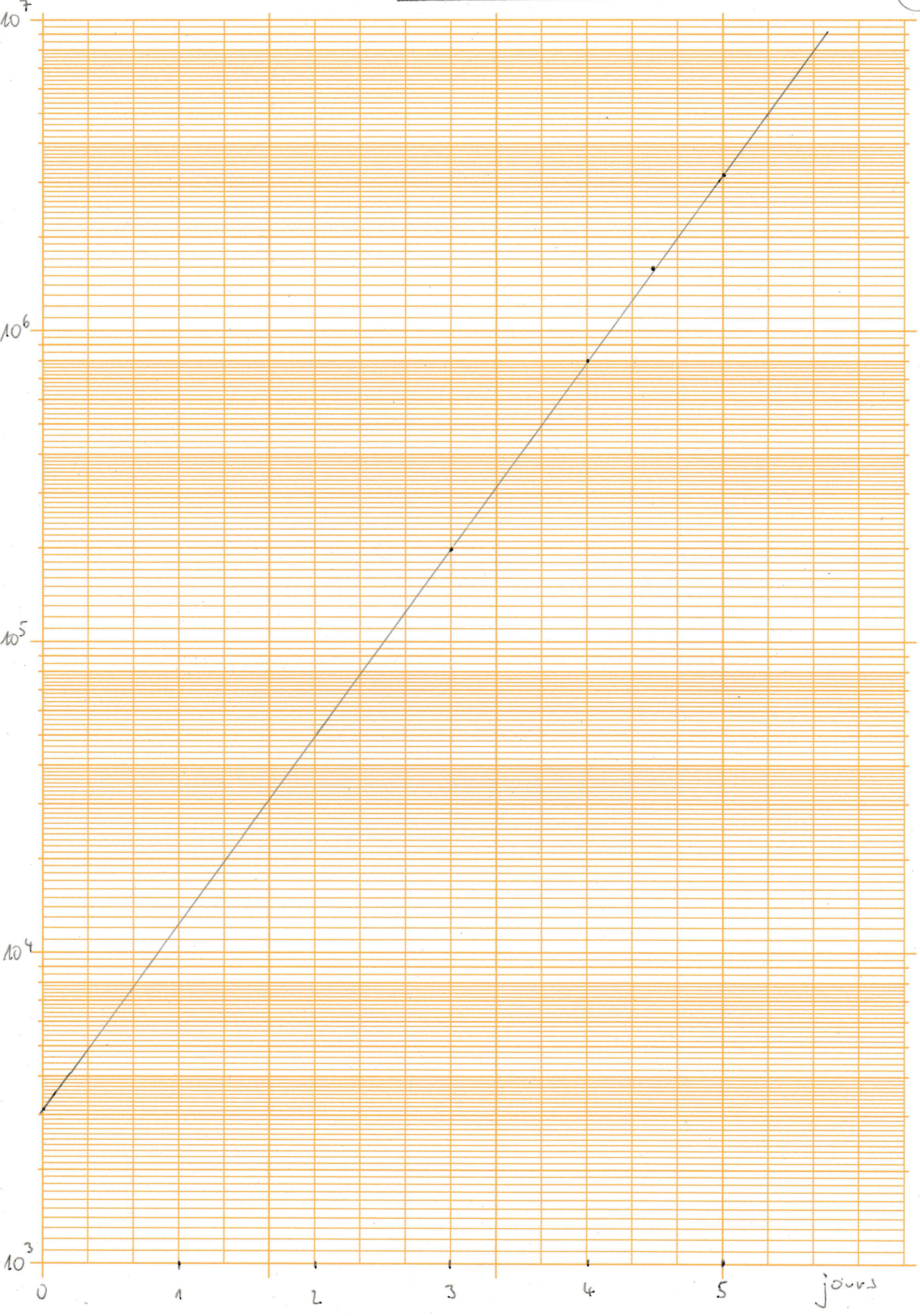
$$\ln(256) = 1,3863\, t \Rightarrow \underline{\underline{t = 4}}$$

c.



Exercice 6.21 c

\# bacteries



Exercice 6.22

$$y = \frac{k}{1+b e^{-cx}}$$

x : temps (jours)

$k = 105$, $c = 1{,}1244$

a. population initiale → poser $x = 0$

↳ $y = 3$

$$3 = \frac{105}{1 + b \cdot \underbrace{e^{-1{,}1244 \cdot 0}}_{=1}} \Rightarrow 3(1+b) = 105$$

$$3 + 3b = 105$$

$$3b = 102$$

$$b = \frac{102}{3} = \underline{\underline{34}}$$

b. $$52 = \frac{105}{1 + 34 \cdot e^{-1{,}1244x}}$$

$$52 + 1768\, e^{-1{,}1244x} = 105$$

$$e^{-1{,}1244x} = \frac{105-52}{1768} \cong 0{,}03$$

$$\underbrace{\ln(e^{-1{,}1244x})}_{=-1{,}1244x} = \ln(0{,}03)$$

$$x = \frac{\ln(0{,}03)}{-1{,}1244} \cong \underline{\underline{3{,}12}} \text{ jours}$$

c. "une longue période de temps" → x tend vers l'infini

$$y = \frac{105}{1 + 34 \cdot \underbrace{e^{-\infty}}_{=0}} = \underline{\underline{105}} \qquad \left(e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0\right)$$

d.

Exercice 6.23

$$f(t) = 200(1-0{,}956 \cdot e^{-0{,}18t}) \qquad t \text{ en années}$$

a. $f(10) = 200 - 191{,}2 \cdot e^{-1{,}8} = \underline{\underline{168{,}4}}$ cm

b. $\overset{1\,m}{100} = 200(1-0{,}956 \cdot e^{-0{,}18t})$

$$\frac{1}{2} = 1 - 0{,}956\, e^{-0{,}18t}$$

$$0{,}956\, e^{-0{,}18t} = \frac{1}{2} \Rightarrow e^{-0{,}18t} = 0{,}523 \Rightarrow t = \frac{\ln(0{,}523)}{-0{,}18} = \underline{\underline{3{,}6}} \text{ ans}$$

c. si $t \to \infty$, $f(t) = 200(1-0{,}956\underbrace{e^{-\infty}}_{=0}) = \underline{\underline{200}}$ cm

Exercice 6.24

$$D = 400 \cdot \log\left(\frac{p}{1-p}\right)$$

a. $\dfrac{D}{400} = \log\left(\dfrac{p}{1-p}\right) \quad \Big| \; 10^{(\,)}$

rappel : $10^{\log(\text{machin})} = \text{machin}$

$$10^{\frac{D}{400}} = \frac{p}{1-p}$$

$$10^{\frac{D}{400}} - p \cdot 10^{\frac{D}{400}} = p \Rightarrow p + p \cdot 10^{\frac{D}{400}} = 10^{\frac{D}{400}}$$

$$p(1+10^{\frac{D}{400}}) = 10^{\frac{D}{400}} \Rightarrow p = \frac{10^{\frac{D}{400}}}{1+10^{\frac{D}{400}}}$$

On peut simplifier l'écriture en amplifiant par $10^{-\frac{D}{400}}$

$$p(D) = \underline{\underline{\frac{1}{10^{-\frac{D}{400}}+1}}}$$

⚠ $D = 1800 - 2005 = -205$

b. joueur 1800 Elo : $1800 + 15\left(0{,}5 - \dfrac{1}{10^{-\frac{-205}{400}}+1}\right) = 1804$

joueur 2005 Elo : $2005 + 15\left(0{,}5 - \dfrac{1}{10^{-\frac{205}{400}}+1}\right) = 2001$

$D = 2005 - 1800 = 205$

Exercice 6.25

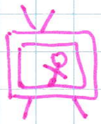
b. température initiale : $T(0) = 4 = 22 - a\,\overbrace{e^{b\cdot 0}}^{1}$

$\Rightarrow 4 = 22 - a \Rightarrow a = \underline{\underline{18}}$

Après une heure : $T(1) = 10 = 22 - 18 \cdot e^{b\cdot 1}$

$\Rightarrow 12 = 18 \cdot e^{b} \Rightarrow e^{b} = \dfrac{2}{3} \Rightarrow b = \ln\left(\dfrac{2}{3}\right) \cong \underline{\underline{-0{,}405}}$

c. $T(3) = 22 - 18 \cdot e^{-0{,}405 \cdot 3} = \underline{\underline{16{,}66^{\circ}}}$



d. $18 = 22 - 18 \cdot e^{-0{,}405 \cdot t}$

$4 = 18 \cdot e^{-0{,}405 t} \Rightarrow e^{-0{,}405 t} = \dfrac{4}{18} \Rightarrow -0{,}405\, t = \ln\left(\dfrac{2}{9}\right)$

$\Rightarrow t = \underline{\underline{3{,}71}}$ heures

Exercice 6.26

2 chiffres

3 chiffres

On va utiliser les log en base 10.

Remarquons que $\log(99) = 1{,}995$, $\log(100) = 2$, $\log(999) = 2{,}999$, etc.

On voit la partie entière donne le nombre de chiffres -1.

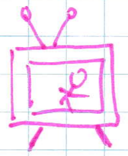
Donc : a) $\log\left(2^{132'049}\right) = 132'049 \cdot \log(2) = 39750{,}71$

$\Rightarrow 2^{132'049}$ a $\underline{\underline{39751}}$ chiffres. ($2^{132\,042} - 1$ aussi puisque $2^{132\,042}$ n'est pas une puissance de 10.)

b) $57\,885\,161 \log(2) = 17'425'169{,}76$

$\Rightarrow 2^{57\,885\,161} - 1$ a $\underline{\underline{17'425'170}}$ chiffres

Exercice 6.27



a. La hauteur se trouve en posant $x = 0$.

$y = -39 \cdot \dfrac{e^{0} + e^{-0}}{2} + 231 = 231 - 39 = 192$ mètres

b. La largeur vaut $2 \times 96 = 192$ mètres, puisque x varie entre -96 et 96.