4. Approximation de fonctions

Exercice 4.1

a. $u_n = \frac{4}{n!}$ $\quad A = \lim_{n\to\infty} \frac{\frac{4}{(n+1)!}}{\frac{4}{n!}} = \lim_{n\to\infty} \frac{\cancel{4}}{(n+1)!} \cdot \frac{n!}{\cancel{4}} = \lim_{n\to\infty} \frac{1}{n+1} = 0$

$\Rightarrow r = \text{"}\frac{1}{0}\text{"} = \infty \quad \Rightarrow$ converge $\forall x$

b. $u_n = 2^n + 1$ $\quad A = \lim_{n\to\infty} \frac{2^{n+1}+1}{2^n+1} \underset{\uparrow}{=} \lim_{n\to\infty} \frac{2^{n+1}\cdot \ln(2)}{2^n \cdot \ln(2)} = 2 \Rightarrow r = \frac{1}{2}$

l'Hôpital

Quand $x = \frac{1}{2}$: $\sum (2^n+1)\cdot\left(\frac{1}{2}\right)^n = \sum \left(1 + \frac{1}{2^n}\right)$ diverge (critère de convergence)

Quand $x = -\frac{1}{2}$: $\sum (2^n+1)\left(-\frac{1}{2}\right)^n = \sum\left(\frac{2^n}{(-2)^n} - \frac{1}{2^n}\right) = \sum\left((-1)^n - \frac{1}{2^n}\right)$ diverge

$\Rightarrow$ converge si $-\frac{1}{2} < x < \frac{1}{2}$

c. $u_n = \frac{1}{n^n}$ $\quad C = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{1}}{\sqrt[n]{n^n}} = \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow r = \infty$

$\Rightarrow$ converge $\forall x$

d. $u_n = \frac{3}{(n+1)^n}$ $\quad C = \lim_{n\to\infty} \sqrt[n]{\frac{3}{(n+1)^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{3}}{n+1} = \text{"}\frac{1}{\infty}\text{"} = 0 \Rightarrow r = \infty$

$\Rightarrow$ converge $\forall x$

e. $u_n = \frac{(-1)^{n+1}}{n}$ $\quad C = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2}}{n+1}}{\frac{(-1)^{n+1}}{n}} \right| = \lim_{n\to\infty} \left| \underbrace{\frac{(-1)^{n+2}}{(-1)^{n+1}}}_{=-1} \cdot \underbrace{\frac{n}{n+1}}_{\to 1} \right| = 1 \Rightarrow r = 1$

Quand $x = -1$ $\quad \sum \frac{(-1)^{n+1}}{n}(-1)^n = \sum \frac{(-1)^{2n+1}}{n} = -\sum \frac{1}{n}$ diverge

Quand $x = 1$ $\quad \sum \frac{(-1)^{n+1}}{n}(1)^n = \sum \frac{(-1)^{n+1}}{n}$ converge (critère des séries alternées)

$\Rightarrow$ converge si $-1 < x \leq 1$

Exercice 4.2

a. $\sum\limits_{n\geq 1} n x^n \Rightarrow u_n = n \qquad A = \lim\limits_{n\to\infty}\left|\frac{n+1}{n}\right| = 1 \Rightarrow r = 1$

Quand $x=-1$: $\sum n(-1)^n$ diverge

Quand $x=1$: $\sum n\cdot 1^n = \sum n$ diverge

$\Rightarrow$ converge si $-1<x<1$

b. $\sum\limits_{n\geq 1} \frac{(-1)^{n+1}}{n} x^n \Rightarrow u_n = \frac{(-1)^{n+1}}{n}$ voir ex 4.1 e ...

c. $\sum\limits_{n\geq 1}\frac{x^n}{n\cdot(n+1)} \Rightarrow u_n=\frac{1}{n(n+1)} \qquad A=\lim\limits_{n\to\infty}\left|\frac{\frac{1}{(n+2)(n+1)}}{\frac{1}{(n+1)\cdot n}}\right| = \lim\limits_{x\to\infty}\frac{\cancel{(n+1)}\cdot n}{(n+2)\cancel{(n+1)}} = 1$

$\Rightarrow r=1$

Quand $x=-1$: $\sum \frac{(-1)^n}{n(n+1)}$ converge (th 3.2)

Quand $x=1$: $\sum \frac{1^n}{n(n+1)} = 1$ (v. p.24)

$\Rightarrow$ converge si $-1\leq x\leq 1$

d. $\sum\limits_{n\geq 1}\frac{(-1)^{n+1}\cdot x^n}{n\cdot 5^n} \Rightarrow u_n=\frac{(-1)^{n+1}}{n\cdot 5^n}$

$$A=\lim_{n\to\infty}\left|\frac{(-1)^{n+2}}{(n+1)5^{n+1}}\cdot\frac{n\cdot 5^n}{(-1)^{n+1}}\right| = \lim_{n\to\infty}\left|\underbrace{\frac{(-1)^{n+2}}{(-1)^{n+1}}}_{-1}\cdot\underbrace{\frac{5^n}{5^{n+1}}}_{\frac{1}{5}}\cdot\underbrace{\frac{n}{n+1}}_{\to 1}\right| = \frac{1}{5} \Rightarrow r=5$$

Quand $x=-5$: $\sum\frac{(-1)^{n+1}\cdot(-5)^n}{n\ 5^n} =$

$\Rightarrow \sum \frac{\overbrace{(-1)^{n+1}\cdot(-1)^n}^{=-1}\ \cancel{5^n}}{n\cdot\cancel{5^n}} = \sum\frac{-1}{n}$ diverge

Quand $x=5$: $\sum\frac{(-1)^{n+1}\ \cancel{5^n}}{n\cdot\cancel{5^n}} = \sum\frac{(-1)^{n+1}}{n}$ converge (th 3.2)

$\Rightarrow$ converge si $-5<x\leq 5$

e. $\sum\limits_{n\geq 1}\frac{x^{2n-2}}{n(n+1)(n+2)} \qquad A=\lim\limits_{n\to\infty}\left|\frac{x^{\overbrace{2(n+1)-2}^{=2n}}}{\cancel{(n+1)}\cancel{(n+2)}(n+3)}\cdot\frac{n\cancel{(n+1)}\cancel{(n+2)}}{x^{2n-2}}\right| =$

$$= \lim_{n\to\infty}\underbrace{\left(\frac{n}{n+3}\right)}_{\to 1}\cdot\underbrace{\frac{x^{2n}}{x^{2n-2}}}_{=\frac{1}{x^{-2}}=x^2} = x^2 \Rightarrow r=\frac{1}{x^2}$$

méthode un peu différente mais le principe est le même

$\Rightarrow$ converge si $-1<x<1$

suite →

Quand $x=-1$: $\sum_{n\geq 1} \frac{(-1)^{2n-2}}{n(n+1)(n+2)} = \sum_{n\geq 1} \frac{1}{n(n+1)(n+2)}$ pair

Quand $x=1$: $\sum_{n\geq 1} \frac{1^{2n-2}}{n(n+1)(n+2)} = \sum_{n\geq 1} \frac{1}{n(n+1)(n+2)}$ pair

$\left\{\begin{array}{l}\text{converge} \\ \text{car } \sum \frac{1}{n(n+1)} \\ \text{converge (p.24)}\end{array}\right.$

$\Rightarrow$ converge si $-1 \leq x \leq 1$

f. $\sum_{n\geq 2} \left(\frac{x}{\ln(n)}\right)^n \Rightarrow u_n = \frac{1}{(\ln(n))^n}$ $\quad \lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln(n))^n}} = \lim_{n\to\infty} \frac{1}{\ln(n)} = 0$

$\Rightarrow r = \infty \Rightarrow$ converge $\forall x$

Exercice 4.3

À la semaine prochaine..