4. Approximation de fonctions

Exercice 4.1

a. $u_n = \frac{4}{n!}$ $\quad A = \lim_{n\to\infty} \frac{\frac{4}{(n+1)!}}{\frac{4}{n!}} = \lim_{n\to\infty} \frac{\cancel{4}}{(n+1)!} \cdot \frac{n!}{\cancel{4}} = \lim_{n\to\infty} \frac{1}{n+1} = 0$

$\Rightarrow r = \text{"}\frac{1}{0}\text{"} = \infty \quad \Rightarrow$ converge $\forall x$

b. $u_n = 2^n + 1$ $\quad A = \lim_{n\to\infty} \frac{2^{n+1}+1}{2^n+1} \underset{\text{l'Hôpital}}{\overset{\uparrow}{=}} \lim_{n\to\infty} \frac{2^{n+1}\cdot \cancel{\ln(2)}}{2^n \cdot \cancel{\ln(2)}} = 2 \Rightarrow r = \frac{1}{2}$

Quand $x = \frac{1}{2}$: $\sum (2^n+1)\cdot\left(\frac{1}{2}\right)^n = \sum\left(1+\frac{1}{2^n}\right)$ diverge (critère de convergence)

Quand $x = -\frac{1}{2}$: $\sum (2^n+1)\left(-\frac{1}{2}\right)^n = \sum\left(\frac{2^n}{(-2)^n} - \frac{1}{2^n}\right) = \sum\left((-1)^n - \frac{1}{2^n}\right)$ diverge

$\Rightarrow$ converge si $-\frac{1}{2} < x < \frac{1}{2}$

c. $u_n = \frac{1}{n^n}$ $\quad C = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{1}}{\sqrt[n]{n^n}} = \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow r = \infty$

$\Rightarrow$ converge $\forall x$

d. $u_n = \frac{3}{(n+1)^n}$ $\quad C = \lim_{n\to\infty} \sqrt[n]{\frac{3}{(n+1)^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{3}}{n+1} = \text{"}\frac{1}{\infty}\text{"} = 0 \Rightarrow r = \infty$

$\Rightarrow$ converge $\forall x$

e. $u_n = \frac{(-1)^{n+1}}{n}$ $\quad C = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2}}{n+1}}{\frac{(-1)^{n+1}}{n}} \right| = \lim_{n\to\infty} \left| \underbrace{\frac{(-1)^{n+2}}{(-1)^{n+1}}}_{=-1} \cdot \underbrace{\frac{n}{n+1}}_{\to 1} \right| = 1 \Rightarrow r = 1$

Quand $x = -1$ $\quad \sum \frac{(-1)^{n+1}}{n}(-1)^n = \sum \frac{(-1)^{2n+1}}{n} = -\sum \frac{1}{n}$ diverge

Quand $x = 1$ $\quad \sum \frac{(-1)^{n+1}}{n}(1)^n = \sum \frac{(-1)^{n+1}}{n}$ converge (critère des séries alternées)

$\Rightarrow$ converge si $-1 < x \leq 1$

Exercice 4.2

a. $\sum_{n\geq 1} n x^n \Rightarrow u_n = n \quad A = \lim_{n\to\infty} \left| \frac{n+1}{n} \right| = 1 \quad \Rightarrow r = 1$

Quand $x = -1$: $\sum n(-1)^n$ diverge

Quand $x = 1$: $\sum n \cdot 1^n = \sum n$ diverge

$\Rightarrow$ converge si $-1 < x < 1$

b. $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} x^n \Rightarrow u_n = \frac{(-1)^{n+1}}{n}$ voir ex 4.1 e ...

c. $\sum_{n\geq 1} \frac{x^n}{n\cdot(n+1)} \Rightarrow u_n = \frac{1}{n(n+1)} \quad A = \lim_{n\to\infty} \left| \frac{\frac{1}{(n+2)(n+1)}}{\frac{1}{(n+1)\cdot n}} \right| = \lim_{x\to\infty} \frac{\cancel{(n+1)}\cdot n}{(n+2)\cancel{(n+1)}} = 1$

$\Rightarrow r = 1$

Quand $x = -1$: $\sum \frac{(-1)^n}{n(n+1)}$ converge (th 3.2)

Quand $x = 1$: $\sum \frac{1^n}{n(n+1)} = 1$ (v. p.24)

$\Rightarrow$ converge si $-1 \leq x \leq 1$

d. $\sum_{n\geq 1} \frac{(-1)^{n+1}\cdot x^n}{n\cdot 5^n} \Rightarrow u_n = \frac{(-1)^{n+1}}{n\cdot 5^n}$

$$A = \lim_{n\to\infty} \left| \frac{(-1)^{n+2}}{(n+1)5^{n+1}} \cdot \frac{n\cdot 5^n}{(-1)^{n+1}} \right| = \lim_{n\to\infty} \left| \underbrace{\frac{(-1)^{n+2}}{(-1)^{n+1}}}_{-1} \cdot \underbrace{\frac{5^n}{5^{n+1}}}_{\frac{1}{5}} \cdot \underbrace{\frac{n}{n+1}}_{\to 1} \right| = \frac{1}{5} \Rightarrow r = 5$$

Quand $x = -5$: $\sum \frac{(-1)^{n+1}\cdot(-5)^n}{n\; 5^n} = \sum \frac{\overbrace{(-1)^{n+1}\cdot(-1)^n}^{=-1}\cancel{5^n}}{n\cdot\cancel{5^n}} = \sum \frac{-1}{n}$ diverge

Quand $x = 5$: $\sum \frac{(-1)^{n+1}\cancel{5^n}}{n\cdot\cancel{5^n}} = \sum \frac{(-1)^{n+1}}{n}$ converge (th 3.2)

$\Rightarrow$ converge si $-5 < x \leq 5$

e. $\sum_{n\geq 1} \frac{x^{2n-2}}{n(n+1)(n+2)} \quad A = \lim_{n\to\infty} \left| \frac{\overbrace{x^{2(n+1)-2}}^{=x^{2n}}}{\cancel{(n+1)}\cancel{(n+2)}(n+3)} \cdot \frac{n\cancel{(n+1)}\cancel{(n+2)}}{x^{2n-2}} \right| =$

méthode un peu différente mais le principe est le même

$$= \lim_{n\to\infty} \underbrace{\left(\frac{n}{n+3}\right)}_{\to 1} \cdot \underbrace{\frac{x^{2n}}{x^{2n-2}}}_{=\frac{1}{x^{-2}}=x^2} = x^2 \Rightarrow r = \frac{1}{x^2}$$

$\Rightarrow$ converge si $-1 < x < 1$

suite $\rightarrow$

Quand $x=-1$: $\displaystyle\sum_{n\geq 1} \frac{(-1)^{2n-2}}{n(n+1)(n+2)}$ pair $= \displaystyle\sum_{n\geq 1} \frac{1}{n(n+1)(n+2)}$

Quand $x=1$: $\displaystyle\sum_{n\geq 1} \frac{1^{2n-2}}{n(n+1)(n+2)}$ pair $= \displaystyle\sum_{n\geq 1} \frac{1}{n(n+1)(n+2)}$

converge car $\displaystyle\sum \frac{1}{n(n+1)}$ converge (p.24)

$\Rightarrow$ converge si $-1 \leq x \leq 1$

f. $\displaystyle\sum_{n\geq 2} \left(\frac{x}{\ln(n)}\right)^n \Rightarrow u_n = \frac{1}{(\ln(n))^n}$ $\displaystyle\lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln(n))^n}} = \lim_{n\to\infty} \frac{1}{\ln(n)} = 0$

$\Rightarrow r = \infty \Rightarrow$ converge $\forall x$

Exercice 4.3

a. $f(x) = \sin(x)$ $\quad f'(x) = \cos(x)$ $\quad f''(x) = -\sin(x)$ $\quad f^{(3)}(x) = -\cos(x)$

$$f(x) = \sin(0) + \frac{\cos(0)}{1!}x - \frac{\sin(0)}{2!}x^2 - \frac{\cos(0)}{3!}x^3 + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

rayon de convergence : $A = \displaystyle\lim_{n\to+\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right|$

$$= \lim_{n\to+\infty} \left| \frac{-x^2}{(2n+3)(2n+2)} \right| \to 0 \cdot x^2$$

converge si $0 \cdot x^2 < 1 \Rightarrow |x| < +\infty$

b. $f(x) = f'(x) = f''(x) = f^{(3)}(x) = \ldots = e^x$

$$f(x) = e^0 + e^0 x + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 + \ldots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$A = \lim_{n\to+\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to+\infty} \left| \frac{x}{n+1} \right| \to 0 \cdot x$$

converge si $0 \cdot x < 1 \Rightarrow |x| < -\infty$

c. $f(x) = \ln(1+x)$ $\quad f'(x) = \frac{1}{1+x}$ $\quad f''(x) = \frac{-1}{(1+x)^2}$ $\quad f^{(3)}(x) = \frac{2}{(1+x)^3}$

$$f^{(4)}(x) = \frac{-6}{(1+x)^4} = \frac{-3!}{(1+x)^4}$$

$$f(x) = \ln(1) + \frac{1}{1+0}x - \frac{1}{2!}\frac{1}{(1+0)^2}x^2 + \frac{1}{3!}\frac{2}{(1+0)^3}x^3 - \frac{1}{4!}\frac{3!}{(1+0)^4}x^4$$

$$= 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$A = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}x^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n+1}x^n}\right| = \lim_{n\to\infty}\left|-x \underbrace{\frac{n}{n+1}}_{\to 1}\right| = x$$

Converge si $|x| < 1$

Exercice 4.4 voir corrigé

Exercice 4.5

a. $f(x) = \ln(2x+5)$ $\quad f'(x) = \frac{2}{2x+5}$ $\quad f''(x) = \frac{-4}{(2x+5)^2}$

$f^{(3)}(x) = \frac{16}{(2x+5)^3}$ $\quad f^{(4)}(x) = \frac{-96}{(2x+5)^4}$ $\quad f^{(5)}(x) = \frac{768}{(2x+5)^5}$

$$f(x) = \ln(5) + \frac{2}{5}x - \frac{2}{25}x^2 + \frac{8}{375}x^3 - \frac{4}{625}x^4 + \frac{32}{15'625}x^5 + R_5(x)$$

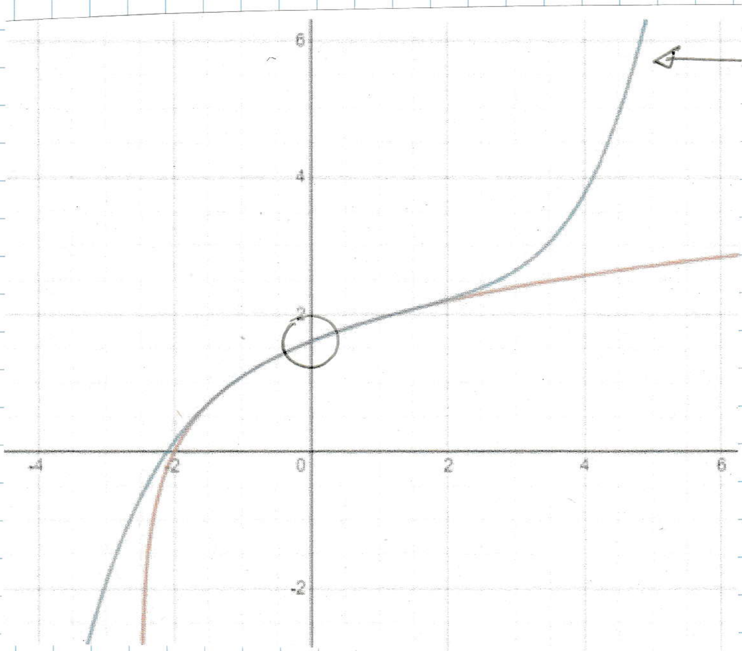
$$f^{(6)}(x) = \frac{-7680}{(2x+5)^6}$$

$$|R_5(x)| \leq \frac{|x|^6}{6!}\, 7680 = \frac{32}{3}x^6$$

M : plus grande valeur possible de $|f^{(6)}(x)|$

dans l'intervalle $[-2 ; 2]$

Ici, c'est quand $x = -2$



$\ln(5) + \frac{2}{5}x - \frac{2}{25}x^2 + \frac{8}{375}x^3 - \frac{4}{625}x^4 + \frac{32}{15625}x^5$

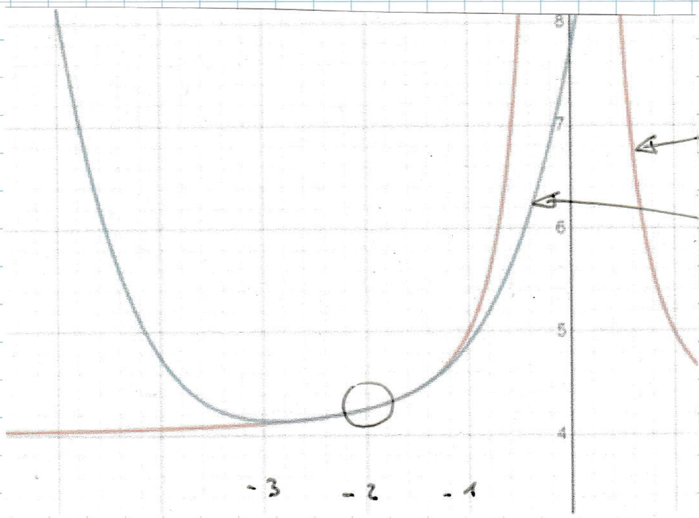
$\ln(2x+5)$

b. $f(x) = \frac{1}{x^2} + 4 \qquad f'(x) = -\frac{2}{x^3} \qquad f''(x) = \frac{6}{x^4} \qquad f^{(3)}(x) = \frac{-24}{x^5} \qquad f^{(4)}(x) = \frac{120}{x^6}$

$f(-2) = \frac{17}{4} \qquad f'(-2) = \frac{1}{4} \qquad f''(-2) = \frac{3}{8} \qquad f^{(3)}(-2) = \frac{3}{4} \qquad f^{(4)}(-2) = \frac{15}{8}$

$$f(x) = \frac{17}{4} + \frac{1}{4}(x+2) + \frac{3}{16}(x+2)^2 + \frac{1}{8}(x+2)^3 + \frac{5}{64}(x+2)^4 + R_4(x)$$

$$f^{(5)}(x) = \frac{-720}{x^7} \Rightarrow R_4(x) \leq \frac{|x+2|^5}{5!} \cdot \underset{|f^{(5)}(-1)|}{720} = 6|x+2|^5$$



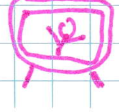
en rouge: $\frac{1}{x^2} + 4$

$\frac{17}{4} + \frac{1}{4}(x+2) + \frac{3}{16}(x+2)^2 + \frac{1}{8}(x+2)^3 + \frac{5}{64}(x+2)^4$

Exercice 4.6

$f(x) = \ln(x) \qquad f'(x) = \frac{1}{x} \qquad f''(x) = -\frac{1}{x^2} \qquad f^{(3)}(x) = \frac{2}{x^3} \qquad f^{(4)}(x) = \frac{-6}{x^4} \qquad f^{(5)}(x) = \frac{24}{x^5}$

$$f(x) = x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}$$ (degré 5)



Exercice 4.7

a. $f(x) = \frac{1}{4x}$ $\quad f'(x) = \frac{-1}{4x^2}$ $\quad f''(x) = \frac{1}{2x^3}$ $\quad f^{(3)}(x) = -\frac{3}{2x^4}$ $\quad f^{(4)}(x) = \frac{6}{x^5}$

$f(5) = \frac{1}{20}$ $\quad f'(5) = \frac{-1}{100}$ $\quad f''(5) = \frac{1}{250}$ $\quad f^{(3)}(5) = -\frac{3}{1250}$

$$f(x) = \frac{1}{20} - \frac{1}{100}(x-5) + \frac{1}{500}(x-5)^2 - \frac{1}{2500}(x-5)^3 + \dots$$

$$= \frac{1}{20}\sum_{k=0}^{\infty}(-1)^k \frac{(x-5)^k}{5^k}$$

b. $f(x) = \cos(2x)$ $\quad f'(x) = -2\sin(2x)$ $\quad f''(x) = -4\cos(2x)$ $\quad f^{(3)}(x) = 8\sin(2x)$

$f(\frac{\pi}{4}) = 0$ $\quad f'(\frac{\pi}{4}) = -2$ $\quad f''(\frac{\pi}{4}) = 0$ $\quad f^{(3)}\left(\frac{\pi}{4}\right) = 8$

$f^{(4)}(x) = 16\cos(2x)$ $\quad f^{(5)}(x) = -32\sin(2x)$

$f^{(4)}\left(\frac{\pi}{4}\right) = 0$ $\quad f^{(5)}\left(\frac{\pi}{4}\right) = -32$

$$f(x) = -2\left(x-\frac{\pi}{4}\right) + \frac{8}{3!}\left(x-\frac{\pi}{4}\right)^3 - \frac{32}{5!}\left(x-\frac{\pi}{4}\right)^5 + \dots$$

$$= \sum_{k=1}^{\infty}(-1)^k \frac{2^{2k-1}}{(2k-1)!}\left(x-\frac{\pi}{4}\right)^{2k-1}$$

Exercice 4.8

À l'ex 4.3b, on a vu que $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

Donc $e^1 = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \dots = 2,71\dots$

Exercice 4.9

À l'ex 4.3c, on a vu que $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots \quad -1 < x \leq 1$

Pour $n = 1$ $\quad \ln(1{,}1) = 0{,}1$ $\quad |R_1| < \frac{0{,}1^2}{2} = 0{,}005$

$n=2: \quad \ln(1{,}1) = 0{,}1 - 0{,}005 = 0{,}095 \qquad |R_2| < \dfrac{0{,}1^3}{3} = 0{,}000\bar{3}$

$n=3: \quad \ln(1{,}1) = 0{,}095 + 0{,}000\bar{3} = 0{,}095\bar{3} \qquad |R_3| < \dfrac{0{,}1^4}{4} = 0{,}000025$

$n=4: \quad \ln(1{,}1) = 0{,}095\bar{3} - 0{,}000025 = 0{,}095308\bar{3} \qquad |R_4| < \dfrac{0{,}5^5}{5} = 0{,}000002$

$\Rightarrow \ln(1{,}1) \cong \underline{\underline{0{,}09531}}$

Ex 4.10

$f(x) = \arcsin(x) \qquad f(0) = 0$

$f'(x) = (1-x^2)^{-\frac{1}{2}} \qquad f'(0) = 1$

$f''(x) = -\frac{1}{2}(1-x^2)^{-\frac{3}{2}} \cdot (-2x) = x(1-x^2)^{-\frac{3}{2}} \qquad f''(0) = 0$

$f^{(3)}(x) = (1-x^2)^{-\frac{5}{2}}(1+2x^2) \qquad f^{(3)}(0) = 1$

$f^{(4)}(x) = (1-x^2)^{-\frac{7}{2}}(6x^3+9x) \qquad f^{(4)}(0) = 0$

$f^{(5)}(x) = (1-x^2)^{-\frac{9}{2}}(24x^4+72x^2+9) \qquad f^{(5)}(0) = 9$

$$\Rightarrow \arcsin(x) = x + \frac{x^3}{3!} + 9\frac{x^5}{5!} + \dots = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \dots$$

voir table ← on aurait dû commencer par là !

$$\Rightarrow \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6} = \frac{1}{2} + \frac{1}{2\cdot 3}\left(\frac{1}{2}\right)^3 + \frac{3}{2\cdot 4\cdot 5}\left(\frac{1}{2}\right)^5 + \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\left(\frac{1}{2}\right)^7 + \dots$$

$$\Rightarrow \frac{\pi}{6} = 0{,}523525 \quad \Rightarrow \pi \cong \boxed{3{,}141}15$$

Ex 4.11

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0}\left(\frac{x}{x} - \frac{x^2}{2x} + \frac{x^3}{3x} - \frac{x^4}{4x} + \dots\right)$$

$$= \lim_{x\to 0}\left(1 - \underbrace{\frac{x}{2}}_{\to 0} + \underbrace{\frac{x^2}{3}}_{\to 0} - \underbrace{\frac{x}{4}}_{\to 0} + \dots\right) = \underline{\underline{1}}$$

Ex 4.12

Par l'absurde : si e est rationnel, alors e s'écrit sous la forme

$$e = \frac{a}{b}, \text{ avec } a \in \mathbb{N} \text{ et } b \in \mathbb{N}^*$$

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} < \frac{a}{b} < 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{2}{n!} \qquad \Big| \cdot n!$$

$$\underbrace{n! + n! + \frac{n!}{2!} + \frac{n!}{3!} + \dots + 1}_{x \in \mathbb{N}} < \frac{n!\,a}{b} < \underbrace{n! + n! + \frac{n!}{2!} + \frac{n!}{3!} + \dots + 2}_{x+1 \in \mathbb{N}}$$

Si $n \to \infty$, $n > b$ et $\frac{n!\,a}{b}$ est un entier.

Donc il existe un entier entre deux entiers consécutifs. WTF?

Absurde. Donc e est irrationnel.